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Quantization of Boson Fields in Quantum Geometry

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It is shown that the introduction of an upper limit on the acceleration of particles provides a natural cutoff on momenta, avoiding the problem of ultraviolet divergencies in local quantum field theory. Such a cutoff turns out to be related to Planck energy.

1. INTRODUCTION

Quantum field theory (QFT) provides the framework of theoretical description of particles and their interactions, the latter ones represented by interaction of several fields of one or several species at the same space–time point. The way quantum fields interact is determined by the symmetries existing in nature, the so-called Gauge Symmetries.

Despite the innumerable theoretical predictions and experimental confirmations of QFT, a peculiar difficulty of this theory is the presence of ultraviolet divergencies, related to the pointlike nature of particles. For overcoming the meaningless infinite expressions, coming from higher order corrections in the perturbative calculations, a renormalization (and regularization) procedure has been developed. In such a way, the infinities are absorbed in a redefinition of a finite number of parameters occurring in the theory.

A more sophisticated theory, where the ultraviolet divergencies are cured in a natural way, is the string theory. Here a fundamental length, the Planck length, is introduced for taking into account the finite extension of particles (Green *et al.*, 1987). Such a minimal observable distance appears also in the framework of quantum gravity, due essentially to the quantum fluctuations of the gravitational field.

The problem of ultraviolet divergencies in local QFT has been recently discussed in the framework of an absolutely different approach (Feoli *et al.*, 1999a). It has been shown that the introduction of an upper limit on the proper acceleration of

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particles could free the quantum theory from divergencies originated by the point-like character of particles or, at least, could reduce the degrees of these divergencies.

Maximal acceleration (MA) of elementary particles is a consequence of quantum geometry theory, proposed by Caianiello in an attempt to unify quantum mechanics and general relativity principles (Caianiello, 1980, 1992). This new geometric formulation of quantum mechanics has been developed by interpreting the quantization as a curvature in the relativistic eight-dimensional space–time tangent bundle $TM_8 = M_4 \otimes TM_4$, that incorporates both the space–time manifold M_4 and the four-velocity space TM_4. The usual position and momentum operators of the Heisenberg algebra are represented as the covariant derivative in TM_8, the quantum commutation relations being treated as the components of the curvature tensor. It is remarkable that the line element in TM_8 intrinsically involves an upper limit on the proper acceleration of the particle (Caianiello, 1980, 1992).

In Caianiello's model, MA is interpreted as mass-dependent, $\mathcal{A}_m = 2mc^3/\hbar$, where m is the mass of the particle, or, as we will do in this paper, a universal constant depending on Planck mass, $\mathcal{A} = m_P c^3/\hbar$, (see also Brandt, 1983, 1984, 1989; Toller, 1990, 1991). It has several implications for the energy spectrum of uniformly accelerated particles (Caianiello *et al.*, 1990), the expansion of the very early universe (Caianiello *et al.*, 1991), the tunneling from nothing (Capozziello *et al.*, 1994), and the mass of the Higgs boson (Kuwata, 1996; Lambiase *et al.*, 1999). It also makes the metric observer-dependent, as conjectured by Gibbons and Hawking (1977), and lead in a natural way to hadronic confinement (Caianiello *et al.*, 1988). MA allows the derivation of the generalized uncertainty principle of string theory (Capozziello *et al.*, 2000a), and its consequences for particles propagating in Schwarzschild-like geometries have been studied in Feoli *et al.* (1999b) and Capozziello *et al.* (2000b). Concrete experimental tests of the consequences of MA have been proposed in Papini *et al.* (1995) and Lambiase *et al.* (1998).

We also recall that MA is the same cutoff on the acceleration required in an ad hoc fashion by Sanchez in order to regularize the entropy and the free energy of quantum strings (De Vega and Sanchez, 1988; Frolov and Sanchez, 1991; Sanchez, 1993), and it is also invoked as a necessary cutoff by McGuigan in the calculation of black hole entropy (McGuigan, 1994).

The aim of this paper is to show that, by quantizing a massless boson field in an accelerating frame (Rindler observer), MA provides a natural cutoff on the momentum that is related to the Planck energy. Such a result is inferred in the framework of the *thermalization theorem*. Essentially, this theorem asserts that a uniformly accelerated particle-detector (with acceleration a) in Minkowski space will be excited by the quantum field whose statistical distribution is spin-dependent. The temperature T of the thermal bath is related to acceleration by the relation (in natural units)

$$T = \frac{a}{2\pi}\cdot \tag{1}$$

If the quantum field is a boson field, the uniformly accelerated observer will detect the usual Bose–Einstein statistical distribution, whereas, for a fermion field, the same observer will detect the usual Fermi–Dirac statistical distribution (Takagi, 1986; Unruh and Wald, 1984).

The layout of this paper is as follows: In Section 2 we shortly recall the quantum corrections to the Rindler geometry induced by MA. Section 3 is devoted to the quantization of a scalar field in the modified Rindler geometry. Conclusions are drawn in Section 4.

2. RINDLER GEOMETRY IN CAIANIELLO'S MODEL

The model proposed by Caianiello for including the effects of MA in dynamics was to enlarge the space–time manifold to an eight-dimensional space–time tangent bundle TM_8. In this way the invariant line element is defined as (Caianiello *et al.*, 1990)

$$d\tilde{s}^2 = g_{AB} dX^A \; dX^B, \quad A, B = 1, \ldots, 8, \tag{2}$$

where the coordinates of TM_8 are

$$X^A = \left(x^\mu ; \frac{1}{\mathcal{A}} \; \frac{dx^\mu}{ds} \right), \quad \mu = 1, \ldots, 4, \tag{3}$$

and

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}, \qquad ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \cdot \tag{4}$$

The metric (2) can be written as

$$d\tilde{s}^2 = g_{\mu\nu} \left(dx^\mu \, dx^\nu + \frac{1}{\mathcal{A}^2} \, d\dot{x}^\mu \, d\dot{x}^\nu \right), \tag{5}$$

where $\dot{x}^\mu = dx^\mu / ds$, and an embedding procedure has been developed to find the effective space–time geometry in which a particle can move when the constraint of a MA is present (Caianiello *et al.*, 1990). In fact, if we find the parametric equations that relate the velocity field $\dot{x}^\mu$ to the first four coordinates x^μ, we can calculate the effective four-dimensional metric on the hypersurface locally embedded in TM_8. This procedure strongly depends on the choice of the velocity field of the particle.

Let us consider a portion of space–time spanned by the world lines of uniformly accelerated observers

$$x = \frac{1}{a} \; \cosh \; a\tau, \qquad x^0 = \frac{1}{a} \; \sinh \; a\tau, \tag{6}$$

obtained by varying a and τ according to the Rindler parameterization $\xi = 1/a$, $\eta = a\tau$ (in what follows we work, for simplicity, with a two-dimensional space–time, parameterized by (η, ξ)). The corresponding velocity field is

$$\dot{x} = \sinh\ \eta, \qquad \dot{x}^0 = \cosh\ \eta, \tag{7}$$

so that, from Eq. (5), the Rindler line element $ds^2 = \xi^2 d\eta^2 - d\xi^2$ is generalized as

$$d\tilde{\tau}^2 = \left(\xi^2 - \frac{1}{\mathcal{A}^2}\right) d\eta^2 - d\xi^2 \cdot \tag{8}$$

We point out that the horizon of this manifold is now given by $\xi = 1/\mathcal{A}$, instead of $\xi = 0$, as in Rindler geometry. It is represented in the (x, x^0) plane by the MA hyperbola $x^2 - x^{0\,2} = \mathcal{A}^{-2}$, corresponding to the world line of a uniformly accelerated particle with constant proper acceleration $a = \mathcal{A}$. The replacing of the light cone by a hyperbola as the boundary of the Rindler space–time, automatically provides the horizon regularization. Finally, another important difference from the usual Rindler geometry is that metric (8) describes a curved manifold with scalar curvature given by

$$R = \frac{2}{\mathcal{A}^2}(\xi^2 - \mathcal{A}^{-2})^{-2}, \tag{9}$$

and diverges for $\xi = \mathcal{A}^{-1}$. Therefore, it is a true physical singularity of the space and cannot be removed by a change of the coordinate. In the next section, we will use the modified Rindler geometry for quantizing a massless scalar field and derive the thermalization theorem.

3. THERMALIZATION THEOREM AND MA

The wave equation for scalar massless particle in curvilinear coordinates is

$$\Box\phi = 0, \tag{10}$$

where $\Box = (-g)^{-1/2}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$ is the D'Alembert operator (Birrell and Davies, 1982). The wave functions are normalized by means of the inner product

$$(\phi_1, \phi_2) = -i\int_\Sigma [\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1]\sqrt{-g_\Sigma(x)}\ d\Sigma^\mu, \tag{11}$$

where both ϕ_1 and ϕ_2 are solutions of Eq. (10), $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future-directed unit versor orthogonal to the spacelike hypersurface Σ and $d\Sigma$ is the volume element in Σ. The hypersurface Σ is taken to be a Cauchy surface in space–time and it is possible to show, using Gauss theorem, that the value of (ϕ_1, ϕ_2) is independent of Σ.

In two-dimensional manifold and with $g_{\mu\nu}$ given by Eq. (8), we have

$$(\phi_1, \phi_2) = -i \int_I [\phi_1 \partial_\eta \phi_2^* - \phi_2^* \partial_\eta \phi_1] g^{\eta\eta} \sqrt{-g} \; d\xi$$
$$= -i \int_I [\phi_1 \partial_\eta \phi_2^* - \phi_2^* \partial_\eta \phi_1] \left(\xi^2 - m_P^{-2}\right)^{-1/2} d\xi, \quad (12)$$

where $I = [1/m_P, \infty[$ (we recall that $\mathcal{A} = m_P$).

In Minkowski space–time, where $g_{\mu\nu} = diag\,(1, -1)$, Eq. (10) becomes

$$\left(\frac{\partial^2}{\partial x^{02}} - \frac{\partial^2}{\partial x^2} \right) \psi(x, x^0) = 0, \quad (13)$$

whose solution is the wave function with positive frequency

$$\psi_k(x^0, x) = \psi_0 \;\; e^{ik(x^0 - x)}, \quad (14)$$

with ψ_0 a normalization factor. A general solution is

$$\psi(x^0, x) = \sum_k [b_k \psi_k(x, x^0) + b_k^\dagger \psi_k^*(x, x^0)], \quad (15)$$

where b_k and $b_k^\dagger$ are interpreted in second quantization as the annihilation and creation operators, respectively. The Fock space of stated is constructed starting from the vacuum state, $|0\rangle_M$.

In the modified Rindler metric (8), Eq. (10) assumes the form

$$\left[\frac{1}{\xi^2 - m_P^{-2}} \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2} - \frac{\xi}{\xi^2 - m_P^{-2}} \frac{\partial}{\partial \xi} \right] \phi(\eta, \xi) = 0. \quad (16)$$

The solution (Abramowitz and Stegun, 1972) is

$$\phi_\omega(\eta, z) = \phi_0 \;\; e^{i\omega\eta} F\left(i\omega, -i\omega; \frac{1}{2}; 1 - z \right), \quad (17)$$

where ϕ_0 is a normalization factor, $z = \frac{1}{2}(m_P \xi + 1)$, and F is the hypergeometric function. A general solution in the accelerated frame can be expanded in terms of positive and negative frequencies

$$\phi(\eta, z^*) = \sum_\omega [a_\omega \phi_\omega(\eta, z^*) + a_\omega^\dagger \phi_\omega^*(\eta, z^*)]. \quad (18)$$

Using this decomposition, the Fock space is constructed from the vacuum state, which is annihilated by the operator a_ω, $a_\omega |0\rangle_R = 0$. The number operator $N_\omega = a_\omega^\dagger a_\omega$ will detect particles when evaluated in Minkowski vacuum. This is described by Bogolubov coefficients connecting the bases of the two frames (inertial and accelerated) in the following way

$$a_\omega = \sum_k [\alpha_{\omega k} b_k + \beta_{\omega k}^* b_k^\dagger], \quad (19)$$

The evidence of thermal radiation turns out for the accelerated observer when the Bogolubov coefficient $\beta_{\omega k}$, given by

$$\beta_{\omega k} = -(\psi_k, \phi^*_\omega) = -i \int_{1/m_P}^{\infty} [\psi_k \partial_\eta \phi_\omega - \phi_\omega \partial_\eta \psi_k] \big(\xi^2 - m_P^{-2}\big)^{-1/2} \, d\xi, \tag{20}$$

is different from zero. Its value is

$$\beta_{\omega k} = -\frac{\pi}{2} \psi_0 \phi_0 \frac{k}{m_P} \;\; e^{-\pi\omega/2} H^{(1)}_{i\omega+1}(k/m_P), \tag{21}$$

and in a similar way, one derives the coefficient $\alpha_{\omega k}$

$$\alpha_{\omega k} = -\frac{\pi}{2} \psi_0 \phi_0 \frac{k}{m_P} \;\; e^{\pi\omega/2} H^{(2)}_{i\omega+1}(k/m_P), \tag{22}$$

where $H^{(1)}_{i\omega+1}(k/m_P)$ and $H^{(2)}_{i\omega+1}(k/m_P)$ are the Hankel's functions. The statistical distribution is obtained squaring the coefficient $\beta_{\omega k}$

$$|\beta_{\omega k}|^2 \; = \frac{\pi^2}{4} (\psi_0 \phi_0)^2 \;\; e^{-\pi\omega} \frac{k^2}{m_P^2} H^{(1)}_{i\omega+1}(k/m_P) H^{(2)}_{-i\omega+1}(k/m_P). \tag{23}$$

Because of the complexity of Eq. (23), its implications in QFT are very difficult to analyze exactly. Nevertheless, some considerations can be done. As is well known, the consequence of combining relativistic quantum mechanics with general relativity is that no measurements can be done at distances smaller than the Planck length and that there are no particles heavier than the Planck mass. Then, a theory valid to the Planck scale has to be valid at any other lower energy scale. One may ignore higher energy phenomena in a low energy theory, but the opposite is not true. At the Planck scale, a theory has to be the *Theory of Everything*, in the sense that there cannot be any theory of particles beyond it. If ultraviolet divergencies appear, there is no way to interpret them as coming from a higher energy scale, as in QFT. Hence, no physical understanding can be given to such ultraviolet infinities; so the theory has to be exactly finite and not renormalizable finite. This is also true in quantum geometry. In fact, it reproduces, in the low energy limit, the usual results of QFT in curved space–time, and in the opposite limit, it leads to ill-defined quantities making the theory inconsistent. As a consequence, an upper limit on momenta related to the Planck energy is naturally recovered, resulting in a finite theory. It is worthwhile to discuss explicitly these two limits.

The Bose–Einstein distribution is recovered from Eq. (23) in the regime of energies that are far from the Planck one. $k/m_P < 1$,

$$|\beta_{\omega k}|^2 \; \sim \frac{1}{e^{\Omega/k_B T} - 1}, \tag{24}$$

where T is the Davies–Unruh temperature, defined in (1) (Birrell and Davies, 1992). In deriving (24) we have substituted the Rindler modes ω with the proper energy

$\Omega = a\omega$ of the Rindler particle seen by the observer (Takagi, 1986). Equation (24) is nothing else but the thermalization theorem. The canonical commutation relations of the Weyl–Heisenberg algebra are also preserved, as we can see from (19), in agreement with the usual interpretation of the annihilation and creation operators.

In the limit $k/m_P \gg 1$, the coefficients $\alpha_{\omega k}$ and $\beta_{\omega k}$ assume the asymptotic form (Abramowitz and Stegun, 1972)

$$\alpha_{\omega k} = -\frac{1}{8}\sqrt{\frac{2}{\pi \omega m_P}}\; e^{-i(k/m_P - 3\pi/4)}, \tag{25}$$

$$\beta_{\omega k} = -\frac{1}{8}\sqrt{\frac{2}{\pi \omega m_P}}\; e^{i(k/m_P - 3\pi/4)}, \tag{26}$$

and using Eq. (19), one finds that, for fixed k, the canonical commutation relation of operators a_ω and $a_\omega^\dagger$ vanishes,

$$[a_\omega, a_{\omega'}^\dagger] = 0, \quad \forall \omega, \omega'. \tag{27}$$

For each mode ω, the Weyl–Heisenberg algebra is not preserved, so a_ω and $a_\omega^\dagger$ cannot be interpreted as annihilation and creation operators in Rindler space–time, making it impossible to build up a Fock space for states. Hence, the constraint on the acceleration, introduced in a quantum theory of particles, implies the existence of a cutoff on momenta that, as discussed before, must be given by the Planck mass m_P for the consistence of the theory.

4. CONCLUSIONS

The regularizing properties of MA, introduced by Caianiello as a consequence of his geometrical interpretation of quantum mechanics, have been discussed in the quantization of a massless scalar field.

We have shown that the consistence of the quantization scheme, carried out in the Minkowski and Rindler space–times and linked by a Bogolubov transformation, requires the introduction of a cutoff on the momentum, related to the MA. The latter one provides a natural length (and energy) scale, so that the ultraviolet divergencies are removed in a natural way, without appealing to the renormalization and regularization techniques of QFT. Such a result is consistent with all theories trying to unify particles and their interactions, including gravity.

From this point of view, quantum geometry is very close to string theory, although the concept of extended nature of particles is introduced in a different way. In quantum geometry, in fact, the finite extension of a particle is provided by MA, whose effects become relevant for high values of the acceleration, hence for high energy regimes, whereas in string theory, it is intrinsically contained in the definition of the action of a particle. Such a strict relation between quantum geometry and string theory has been recently discussed also in the paper by Capozziello *et al.* (2000a), in which the generalized uncertainty principle of strings has been derived from the canonical commutation relations in a *curved* space–time induced by the existence of an upper limit on the acceleration of particles.

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